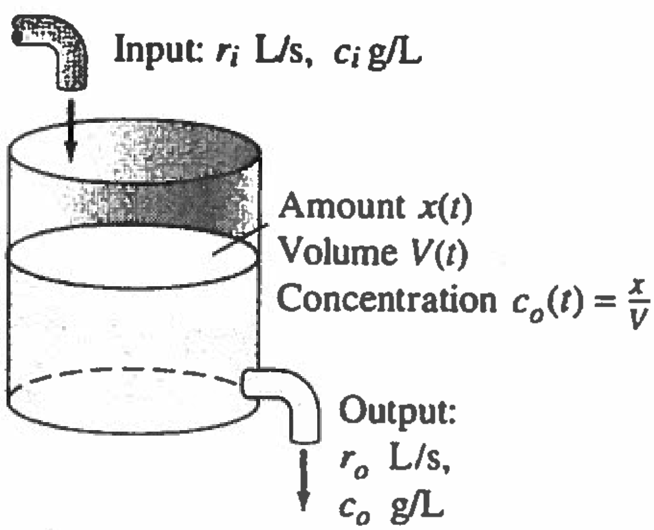
1.5: Linear First-Order Equations
Mixture Problems

We look at our first application of linear first-order equations. Consider a tank containing a solution- a mixture of solute and solvent - such as salt dissolved in water. There is both an inflow and an outflow and we wish to compute the amount of solute $x(t)$ (or in many instance, the percentage of solute $x(t)/V(t)$) in the tank at a given time t.



The above diagram yields the differential equation for mixture problems

$$\frac{dx}{dt} = r_i c_i - \frac{r_0}{V} x. \tag{1}$$

Example 1. Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350 km^3 per year. Suppose that at the time $t = 0$(years), the pollutant concentration of Lake Erie is five times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

$V(t) = 480 \text{ Km}^3$

$r_i = r_o = 350 \text{ Km}^3/\text{yr}$

$\frac{x(0)}{480} = 5 \cdot c_i$

or $x(0) = 5 \cdot c_i \cdot 480$

Then $\frac{dx}{dt} = 350 \cdot c_i - \frac{350 \cdot x}{480}$ $(x(0) = 2400 c_i)$

Or $\frac{dx}{dt} + \frac{35}{48} x = 350 \cdot c_i$

Solve to get $x(t) = 480 c_i + 1920 c_i e^{-35t/48}$

Then $x(t) = 960 c_i$ when

$t = \frac{48}{35} \ln 4 \approx 1.901$ years

Example 2. A tank contains 1000 liters of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture- kept uniform by stirring- is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

$V = 1000$ L

$x(0) = x_0 = 100$ Kg

$r_i = 5$ L/s $= r_o$

$c_i = 0$ Kg/L

$c_o = \frac{x}{V}$ Kg/L

From (1), $\frac{dx}{dt} = 0 - \frac{5x}{1000} = -\frac{1}{200}x.$

So $\frac{dx}{dt} + \frac{x}{200} = 0.$

$\rho(t) = e^{\int \frac{1}{200} dt} = e^{t/200}$

So $x(t) = e^{-t/200}\left[100 + \int_0^t 0\, d\ast\right]$

$= 100 e^{-t/200}$. Then $x(t) = 10$ when $t = -200 \ln(10)$

≈ 7 min 41s.

$=$

461 s.

Exercise 1. A 120-gallon tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the well-stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt does the tank contain when it is full?

120 gal tank

$V_0 = 90$ gal

$x_0 = 90$ lb

$c_i = 2$ lb/gallon

$r_i = 4$ gal/min

$r_o = 3$ gal/min

$V(t) = 90 + t$ gallons

$c_o = x/V$ lb/gal

From (1), $\frac{dx}{dt} = 8 - \frac{3}{90+t}x.$

So $\frac{dx}{dt} + \frac{3}{90+t}x = 8.$

$\rho(t) = e^{3\ln(90+t)} = (90+t)^3$

So $x(t) = (90+t)^{-3}\left[90 + \int_0^t (90+x)^3 \cdot 8\, dx\right]$

$= (90+t)^{-3}\left[90 + 2(90+t)^4\right]$

Tank is full at $t = 30$ min, so $x(30) \approx 202$ lbs of salt

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Homework. 31-41 (odd)